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March 24, 1969

Ms. Winnie Morgan
Acquisitions and Dissemination Branch USI
National Aeronautics and Space Administration
Washington, D. C. 20546

Re: NGR 44-007-028
SMU No. 83-21

Dear Winnie:

In line with our telephone conversation this morning, you will find attached a copy of the report of Dr. John Walsh's dated September 1968 entitled "Chance of All Success for Independent Binomial Events with Large Success Probabilities."

Sincerely,

Mildred

Mrs. Louis E. Haenel

CHANCE OF ALL SUCCESS FOR INDEPENDENT BINOMIAL EVENTS WITH LARGE SUCCESS PROBABILITIES

John E. Walsh
Southern Methodist University*

ABSTRACT

Consider reliability situations where all of n independent steps or operations must be satisfactory (successes) if the overall operation is to be satisfactory. The interest is in reliability levels that are at least moderately high. That is, the probability of obtaining successes for all steps and operations is at least moderately near unity. An approximate expression, also sharp upper and lower bounds, are developed for the probability of all successes. These results are applicable when $n(1 - p) < 1$, where p is the arithmetic average of the n probabilities of success for the steps and operations. The approximate expression is near the bounds when $n(1 - p) \leq .25$ and the relative error is less than one percent when $n(1 - p) \leq .17$; then, the true probability is at least .75 and at least .83, respectively. These results depend only on n and p. They are applicable for all $n \geq 1$ and any set of success probabilities. When $n(1 - p) \leq .2$, the results can be rather accurately expressed in terms of $n(1 - p)$. This material is, of course, also applicable to the probability of all failures when the probability of failure (not success) is large for each of the n independent binomial events.

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INTRODUCTION AND RESULTS

Consider n independent binomial events, where success and failure are the possible outcomes for an event. There is sometimes interest in whether all the events are successes, or whether all the events are failures. As an example, some reliability situations are such that overall operation is satisfactory if and only if all of n independent steps or operations are satisfactory (successes). As another example, there are situations where occurrence of at least one success is enough to assure the overall success of a project (for example, in research efforts). Of course, the probability of at least one success is unity minus the probability of all failures.

High reliabilities (say, probability of overall success is at least .8) are often of interest. Also, research projects where the probability of at least one success is too small (say, $\leq P$, where $P \leq .2$) should likely not be undertaken. These are the ranges of probality levels for which approximate values and bounds are developed. That is, the principal interest is in cases where the probability of all successes is at least .8 , or cases where the probability of all failures is at least .8 . Only the probability of all successes is considered in the remaining material. However, corresponding results are obtained for all failures by exchanging the roles of success and failure.

Let p denote the arithmetic average of the success probabilities for the individual events. When $n(1 - p) < 1$, sharp upper and lower bounds for P(all successes) are provided by

$$1 - n(1 - p) \leq P(\text{all successes}) \leq p^n .$$

For $n(1 - p) \leq .25$, the value of p^n approximately equals

$$1 - n(1 - p) + (\tfrac{1}{2})(1 - p)^2$$

and is less than this value. Thus, for $n(1 - p) \leq .25$, this approximate expression can be used as an approximately sharp upper bound for P(all successes). The arithmetic average of the lower bound and this upper bound is

$$1 - n(1 - p) + (\tfrac{1}{4})(1 - p)^2 = [1 - (\tfrac{1}{2})n(1 - p)]^2 ,$$

which is the approximate expression for P(all successes).

It is to be noted that exact values for P(all successes) could be obtained in terms of the geometric mean of the success probabilities of the individual events. That is, P(all successes) equals this geometric mean raised to the n-th power. However, the multiplicative character of the geometric mean makes it substantially less convenient for use than p (with respect to estimation of its value, etc.) except when it can be approximately expressed in terms of p. For example, the observed number of successes for the n events in an unbiased estimate of p. Thus, past data on similar sets of events can be directly used to estimate p.

The next and final section contains derivations for the results stated above.

DERIVATIONS

Let p_i denote the probability of success for the i-th binomial event $(i = 1, \ldots, n)$. Since the geometric mean of the p_i is at most equal to their arithmetic mean, it follows that P(all successes) is at most equal to p^n, with equality possible when all the p_i are equal.

Now, consider derivation of the sharp lower bound, The value of P(all successes) is

$$\prod_{i=1}^{n} p_i = \exp\left\{\sum_{i=1}^{n} \log_e[1 - (1 - p_i)]\right\}$$

$$= \exp\left[-\sum_{i=1}^{n}\sum_{j=1}^{\infty} (1 - p_i)^j/j\right]$$

$$= \exp\left[-\sum_{i=1}^{\infty} j^{-i}\sum_{k=0}^{j}\binom{j}{k}(1 - p_i)^{j-k}\sum_{i=1}^{n}(p - p_i)^k\right]$$

For $n(1 - p) < 1$ and $k \geq 2$, $\sum_{i=1}^{n}(p - p_i)^k$ is largest when all but one of the p_i are unity and the other one is such that their arithmetic average is p. This implies that the other p_i is $1 - n(1 - p)$, so that the maximum is

$$\sum_{i=1}^{n}(p - p_i)^k = (n - 1)^k(1 - p)^k + (-1)^k(n - 1)(1 - p)^k .$$

Thus since

$$\sum_{k=o}^{j}\binom{j}{k}[(n - 1)^k + (-1)^k(n - 1)] = [(n - 1) + 1]^j + (n - 1)(1 - 1)^j ,$$

which equals n^j , P(all successes) is at least equal to

$$\exp\left\{-\sum_{j=1}^{\infty} j^{-1}(1 - p)^j\sum_{k=0}^{j}\binom{j}{k}[(n - 1)^k + (-1)^k(n - 1)]\right\}$$

$$= \exp\left\{-\sum_{j=1}^{\infty} j^{-1}[n(1 - p)]^j\right\}$$

$$= \exp\left\{\log_e[1 - n(1 - p)]\right\} = 1 - n(1 - p) ,$$

with equality possible.

Alternate proofs for both bounds, also requiring some effort, could be based on the results of (Hoeffding, 1956).

REFERENCE

Hoeffding, Wassily, "On the distribution of the number of successes in independent trials," Annals of Math. Stat., Vol. 27 (1956), pp. 713-721.